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LETTER TO THE EDITOR

Box and ball system with a carrier and ultradiscrete modified KdV equation

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Abstract. A new soliton cellular automaton is proposed. It is defined by an array of an infinite number of boxes, a finite number of balls and a carrier of balls. Moreover, it reduces to a discrete equation obtained from the discrete modified Korteweg–de Vries equation through a limit. An algebraic expression of soliton solutions is also proposed.

In 1990, Takahashi and Satsuma [1] proposed a soliton cellular automaton (SCA). Its state is defined by using an infinite array of boxes and a finite number of balls. Therefore, the SCA is now called a 'box and ball system'(BBS). The time evolution rule is defined by the following equation;

$$T_j^{t+1} = \min\left(L - T_j^t, \sum_{i=-\infty}^{j-1} T_i^t - \sum_{i=-\infty}^{j-1} T_i^{t+1}\right) \tag{1}$$

where T_j^t is a number of balls in jth box at time t and L means every box holds L balls at most. (Though the box capacity L is restricted to one in the original version of the BBS in [1], we can extend the system to the one with boxes of capacity more than one [2, 3].) The remarkable feature of the system is the existence of N-soliton solutions and an infinite number of conserved quantities [4].

Recently, Tokihiro *et al* [5] including us, have revealed the algebraic properties of the BBS by finding the direct relation to discrete soliton equations. The key is the following identity;

$$\lim_{\varepsilon\to+0} \varepsilon \log(\mathrm{e}^{\frac{A}{\varepsilon}} + \mathrm{e}^{\frac{B}{\varepsilon}} + \cdots) = \max(A, B, \ldots). \tag{2}$$

Using this identity, the discrete Lotka–Volterra (d-LV) equation [6];

$$\frac{w_j^{t+1}}{w_j^t} = \frac{1 + \delta w_{j-1}^t}{1 + \delta w_{j+1}^{t+1}}, \tag{3}$$

reduces to the ultradiscrete Lotka–Volterra (u-LV) equation;

$$W_j^{t+1} - W_j^t = \max(0, W_{j-1}^t - L) - \max(0, W_{j+1}^{t+1} - L) \tag{4}$$

if we take $w_j^t = \exp(W_j^t/\varepsilon)$, $\delta = \exp(-L/\varepsilon)$ and a limit $\varepsilon \to +0$. Note that if the parameter L and initial W are all integer, W_j^t for any j and t is always integer.

If we define $\tilde{W}_j^t$ by

$$\tilde{W}_j^t = \sum_{i=-\infty}^{j} (T_{i+1}^t - T_i^{t+1}) \tag{5}$$

and introduce a transformation of coordinates $\tilde{W}_j^t = W_{j-t}^j$, then we can derive equation (4) from equation (1). Thus we can see solutions of the BBS can be expressed by those of u-LV equation. Indeed, N-soliton solutions of the BBS can be derived from those of u-LV equation [5]. The discretization procedure described above is called 'ultradiscretization' and several ultradiscrete equations are successfully derived from difference equations preserving their algebraic properties [7–9].

Tsujimoto and Hirota [10] proposed a discrete version of modified Korteweg–de Vries (d-mKdV) equation;

$$\frac{v_j^{t+1}(1+\delta v_{j+1}^{t+1})}{1+av_j^{t+1}} = \frac{v_j^t(1+\delta v_{j-1}^t)}{1+av_j^t} \tag{6}$$

where δ and a are parameter constants and j and t are integer variables. If we define a new variable $r_n(t)$ by $v_n^t = r_n(-\delta t)$ and take a limit $\delta \to 0$, equation (6) reduces to the following modified version of Lotka–Volterra equation;

$$\dot{r_j} = r_j(1+ar_j)(r_{j+1} - r_{j-1}). \tag{7}$$

Moreover, if we define $s(x,t)$ by $r_j(t) = -\frac{1}{2a} + \sqrt{-1}\varepsilon s((j-\frac{1}{2a}t)\varepsilon, \frac{\varepsilon^3}{3}t)$ and take a limit $\varepsilon \to 0$, then equation (7) reduces to the following modified Korteweg–de Vries (mKdV) equation;

$$s_t + 6as^2 s_x + \frac{1}{4a} s_{xxx} = 0. \tag{8}$$

Maruno *et al* [11] showed that equation (6) can be bilinearized and has N-soliton solution with a Casorati determinant. Thus, equation (6) is a fully discrete soliton equation analogous to the continuous mKdV equation (8).

In this letter, we show that the d-mKdV equation (6) can reduce to an ultradiscrete mKdV (u-mKdV) equation under appropriate transformations of variables and the limit (2). Then, we show that the u-mKdV equation is related to an extended version of BBS introducing a carrier of balls. Finally, we discuss a structure of N-soliton solutions of the system.

First, we derive the u-mKdV equation from the d-mKdV equation. Introducing a variable $\tilde{v}_j^t \equiv v_j^t/(1+av_j^t)$, equation (6) is rewritten as

$$\tilde{v}_j^{t+1} \frac{1+(\delta - a)\tilde{v}_{j+1}^{t+1}}{1-a\tilde{v}_{j+1}^{t+1}} = \tilde{v}_j^t \frac{1+(\delta - a)\tilde{v}_{j-1}^t}{1-a\tilde{v}_{j-1}^t}. \tag{9}$$

Then, introducing another variable V_j^t by $\tilde{v}_j^t = \exp(V_j^t/\varepsilon)$ and taking $\delta = \exp(-L/\varepsilon)$ and $a = -\exp(-M/\varepsilon)$, equation (9) reduces to

$$V_j^{t+1} + \varepsilon \log \frac{1+(\mathrm{e}^{-L/\varepsilon} + \mathrm{e}^{-M/\varepsilon})\mathrm{e}^{V_{j+1}^{t+1}/\varepsilon}}{1+\mathrm{e}^{(V_{j+1}^{t+1}-M)/\varepsilon}} = V_j^t + \varepsilon \log \frac{1+(\mathrm{e}^{-L/\varepsilon} + \mathrm{e}^{-M/\varepsilon})\mathrm{e}^{V_{j-1}^{t}/\varepsilon}}{1+\mathrm{e}^{(V_{j-1}^{t}-M)/\varepsilon}}. \tag{10}$$

If $L \geqslant M$, we obtain a trivial equation from equation (10) under a limit $\varepsilon \to +0$. Therefore, we consider the $L < M$ case. Taking $\varepsilon \to +0$, then

$$\begin{aligned} &V_j^{t+1} + \max(0, V_{j+1}^{t+1} - L) - \max(0, V_{j+1}^{t+1} - M) \\ &\qquad = V_j^t + \max(0, V_{j-1}^t - L) - \max(0, V_{j-1}^t - M) \end{aligned} \tag{11}$$

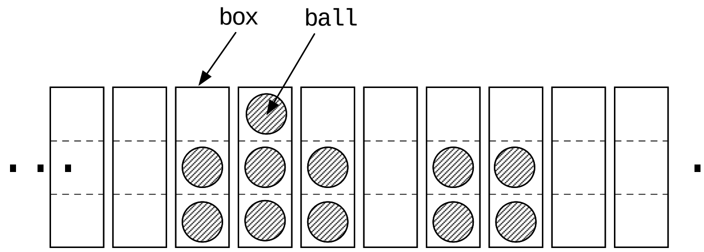

Figure 1. An example of a state for $L = 3$.

through the limit (2).

If we take $M \to \infty$, the last terms of both sides of equation (11) disappear and equation (11) becomes u-LV equation (4). This corresponds to the relation between the d-mKdV equation (6) and the d-LV equation (3) when we take $a = 0$. If L, M and initial V are all integer, V_j^t for any j and t is always integer. We call equation (11) the 'ultradiscrete modified KdV' (u-mKdV) equation.

Next, we define the 'box and ball system with a carrier' (BBSC) and show its evolution rule is derived from the u-mKdV equation. Prepare an array of an infinite number of boxes and a finite number of balls. Assume that all balls are the same, that is, they cannot be distinguished from each other. All boxes are also the same and each box holds L balls at most. A 'state' is defined by putting balls into boxes appropriately. Therefore, any state can be distinguished by the number of balls and the distribution of balls in the array of boxes. Figure 1 shows an example of a state for $L = 3$. Note that the array of boxes is fixed in space and we can identify every box by integer site number j increasing from left to right.

We assume any state can evolve into another state from integer time t to $t+1$. In order to define the evolution rule, prepare a 'carrier' of balls. Assume that the carrier can carry M balls at most. From t to $t+1$, the carrier moves from the $-\infty$ site to the ∞ site and passes each box from left to right. While the carrier passes the jth box, the following action occurs. Assume that the carrier carries m $(0 \leqslant m \leqslant M)$ balls before it passes the jth box. Also assume that there are ℓ $(0 \leqslant \ell \leqslant L)$ balls in the jth box. There are vacant spaces of $M-m$ balls in the carrier and those of $L-\ell$ balls in the box. Then, when the carrier passes the box, the carrier puts $\min(m, L-\ell)$ balls into the box and gets $\min(\ell, M-m)$ balls from the box. In other words, the carrier puts as many balls into the box as possible and simultaneously obtains as many balls from the box as possible. The action of the carrier is illustrated in figure 2.

According to the above rule, the number of balls in the jth box changes from ℓ to $\ell + \min(m, L-\ell) - \min(\ell, M-m) = \min(m, L-\ell) + \max(0, \ell + m - M)$. (Note the identity $-\min(A, B) = \max(-A, -B)$.) Finally, if U_j^t denotes the number of balls in the jth box at time t, an evolution equation to U_j^t is

$$U_j^{t+1} = \min\left(L - U_j^t, \sum_{i=-\infty}^{j-1} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1}\right) + \max\left(0, \sum_{i=-\infty}^{j} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1} - M\right). \tag{12}$$

Note that $U_j^t \to 0$ $(j \to \pm\infty)$ because the total number of balls is finite and that the carrier carries $\sum_{i=-\infty}^{j-1} U_i^t - \sum_{i=-\infty}^{j-1} U_i^{t+1}$ balls just before passing the jth box. All dependent and independent variables of equation (12) are integer and the dependent variable U always

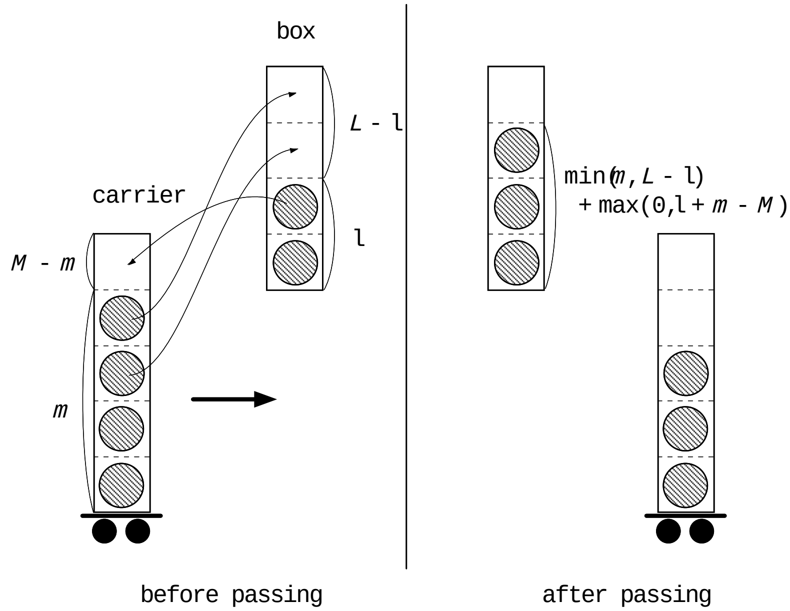

Figure 2. Action of carrier while passing a box.

```
t =  0  : . . . . . 2 3 2 . 2 2 . . . . . . .
     1  : . . . . . . . 3 3 1 1 3 . . . . . .
     2  : . . . . . . . . 1 3 2 . 3 2 . . . .
     3  : . . . . . . . . . . 2 3 . 2 3 1 . .
     4  : . . . . . . . . . . . . 3 1 1 3 3 .
     5  : . . . . . . . . . . . . . 2 2 . 2 3
     6  : . . . . . . . . . . . . . . 1 3 . .
```

Figure 3. Evolution of the state of figure 1 for $L = 3$ and $M = 5$. Each number denotes the number of balls in a box and '.' denotes an empty box.

satisfies $0 \leqslant U \leqslant L$. Therefore, the BBSC can be considered to be a cellular automaton.

Figure 3 shows an evolution of the state of figure 1 for $L = 3$ and $M = 5$. In this figure, each number denotes the number of balls in a box and '.' denotes an empty box. Let us define S_j^t by

$$S_j^t = \sum_{i=-\infty}^{j} U_i^t. \tag{13}$$

Using $U_j^t = S_j^t - S_{j-1}^t$ and equation (12), we can derive

$$S_{j+1}^{t+1} - S_j^t = -\max(0, S_{j+1}^t - S_j^{t+1} - L) + \max(0, S_{j+1}^t - S_j^{t+1} - M). \tag{14}$$

In the derivation, we use the identity $\max(A, B) = A + \max(0, B - A)$. Moreover, if we introduce $\tilde{V}_j^t = S_{j+1}^t - S_j^{t+1}$, then $\tilde{V}_j^t$ satisfies

$$\begin{aligned} &\tilde{V}_{j+1}^{t+1} + \max(0, \tilde{V}_{j+1}^t - L) - \max(0, \tilde{V}_{j+1}^t - M) \\ &\qquad = \tilde{V}_j^t + \max(0, \tilde{V}_j^{t+1} - L) - \max(0, \tilde{V}_j^{t+1} - M). \end{aligned} \tag{15}$$

```
t = 0 : ...11111.....11.....1...........
    1 : ......11111....11....1.........
    2 : .........11111...11...1........
    3 : ............11111..11..1.......
    4 : ...............1111..11.11.....
    5 : ..................111..1.1111..
    6 : .....................11.1..11111
    7 : .......................1.11...11
    8 : ........................1..11..
    9 : .........................1...11
   10 : ..........................1....1
   11 : ...........................1....
```

(a)

```
t =  0 : ...13332..23...13...2.........
     1 : .....13332..32...31..2........
     2 : .......13332.131..22..2.......
     3 : .........13332.23..13..2......
     4 : ...........13331.33..31.2.....
     5 : .............1333.133.22.2....
     6 : ...............1332.2311313...
     7 : .................1331.32.3133.
     8 : ...................133.13.2133
     9 : .....................132.312.2
    10 : .......................13.2131
    11 : .........................312.2
    12 : ..........................213.
    13 : ...........................2.3
    14 : ............................2.
    15 : .............................2
```

(b)

Figure 4. Examples of evolution. (*a*) $L = 1$ and $M = 3$, (*b*) $L = 3$ and $M = 6$.

If we introduce a coordinates transformation, $\tilde{V}_j^t = V_{j-t}^j$, then V_j^t satisfies equation (11). Therefore, we can conclude that the BBSC (equation (12)) reduces to the u-mKdV equation (11) through transformation of variables and coordinates.

Next, we discuss the structure of basic solutions to the BBSC. Figures 4(*a*) and (*b*) shows examples of the evolution of a state of the BBSC. We can observe groups of neighbouring balls separated by empty boxes at every time. Let us call each group a 'ball group'. Moreover, let us define the 'size' of a group by the number of balls included.

Figure 4(*a*) shows the following. For $t \leqslant 3$, there are three ball groups of size 5, 2 and 1, respectively. For $t = 4 \sim 6$, they interact with each other. For $t \geqslant 7$, again three ball groups of the same size appear and they never interact. After the interaction, a shift of orbit occurs for each ball group. We can observe similar phenomena in figure 4(*b*). In the figure, there are four ball groups of size 12, 5, 4 and 2, respectively. Note that we identify a ball group only its size, not its shape. For example, the ball group '23' at $t = 0$, '32' at $t = 1$ and '131' at $t = 2$ in figure 4(*b*) are an identical group. After the interaction, four ball groups reappear and their sizes are the same as those before interaction. Figures 4(*a*) and (b) imply that the BBSC is a soliton system. Despite the interaction, every ball group preserves its own size and speed. Therefore, we can consider each ball group is a soliton.

The most simple solution is a 1-soliton solution. A general expression of a 1-soliton

```
t =  0 : ...144442..........
     1 : .....244441........
     2 : .......34444.......
     3 : .........44443.....
     4 : ..........144442...
     5 : ............244441.
     6 : ..............34444
     7 : ................444
     8 : .................14
```

Figure 5. Example of a 1-soliton for $L = 4$ and $M = 7$.

solution is

$$U_j^t = f_{j+1}^t - f_j^t - g_{j+1}^t + g_j^t \tag{16}$$

with

$$f_j^t = \max(0, kj - \omega t - \xi^0)$$
$$g_j^t = \max(0, kj - \omega t - \xi^0 - n)$$

where n is the size of a ball group, ξ^0 is an initial phase, $k = \min(n, L)$, and $\omega = \min(n, M)$. In figure 5, we show an example of a 1-soliton solution in the case of $L = 4$, $M = 7$, $n = 19$, $\xi^0 = 3$. Since site number j is not specified explicitly in this figure, ξ^0 has a freedom of additional constant.

Let us define a speed of a soliton (ball group) by an average number of boxes which the soliton passes per unit time. Then, a speed of a soliton of size n is $\min(n, M)/\min(n, L)$. Therefore, the maximum speed is M/L. This is a remarkable feature of the BBSC distinguishable from the BBS, because the speed of a soliton of the BBS is unbounded.

The general expression of an N-soliton solution is

$$U_j^t = f_{j+1}^t - f_j^t - g_{j+1}^t + g_j^t \tag{17}$$

with

$$f_j^t = \max_{\mu_i=0,1}\left[\sum_{i=1}^{N}\mu_i\xi_i - \sum_{i<i'}^{(N)}\mu_i\mu_{i'}a_{ii'}\right]$$
$$g_j^t = \max_{\mu_i=0,1}\left[\sum_{i=1}^{N}\mu_i(\xi_i - n_i) - \sum_{i<i'}^{(N)}\mu_i\mu_{i'}a_{ii'}\right]$$

where

$$\xi_i = k_i j - \omega_i t - \xi_i^0$$
$$a_{ii'} = 2\min(n_i, n_{i'})$$
$$k_i = \min(n_i, L)$$
$$\omega_i = \min(n_i, M).$$

Here n_i and ξ_i^0 are a size and an initial phase of each soliton, respectively. $\max_{\mu_i=0,1}[X(\mu_i)]$ denotes the maximum value in 2^N possible values of $X(\mu_i)$ obtained by replacing each μ_i by 0 or 1. $\sum_{i<i'}^{(N)}$ denotes the summation over all possible pairs chosen from N elements.

Note that we derived the above expression of the solution empirically and cannot yet prove it is truly a general expression. However, we confirmed the expression numerically for a wide range of initial data. We obtain the solutions in figure 4 by setting $N = 3$,

$n_1 = 5$, $n_2 = 2$, $n_3 = 1$, $\xi_1^0 = 0$, $\xi_2^0 = 6$, $\xi_3^0 = 13$ (figure 4(*a*)) and $N = 4$, $n_1 = 12$, $n_2 = 5$, $n_3 = 4$, $n_4 = 2$, $\xi_1^0 = 2$, $\xi_2^0 = 12$, $\xi_3^0 = 22$, $\xi_4^0 = 22$ (figure 4(*b*)) in the above expression.

Finally, we give concluding remarks. We proposed a new soliton system, the BBSC. This system is an extended system to the BBS in [1] and can reduce to the u-mKdV equation (11) newly obtained from the d-mKdV equation (6). Moreover, we proposed a general expression of soliton solutions to the BBSC. However, this expression is derived empirically. In [11], algebraic expression of N-soliton solution to the d-mKdV equation is shown. Therefore, it may be possible to derive solutions to the u-mKdV equation from those to the d-mKdV equation using the limit (2). Such a derivation is not automatic and we have not yet succeeded. This is a problem to be solved in the future.

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